

LOWER BOUND OF THE ASYMPTOTIC COMPLEXITY OF SELF-SIMILAR FRACTAL GRAPHS

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ABSTRACT. We study the asymptotic complexity constant of the sequence of approximating graphs to a fully symmetric self-similar structure on a finitely ramified fractal K . We show how full symmetry implies existence of the asymptotic complexity constant and obtain a sharp lower bound thereby answering two conjectures by Anema [1].

1. INTRODUCTION.

An important research topic in graph theory is the enumeration of spanning trees. In recent works, various authors have studied the number of spanning trees on so-called fractal graphs, which are graphs approximating a finitely ramified self-similar set (cf. [4, 5, 6, 18, 20]).

In [1], a different methodology from the previous works is given for calculating the number of spanning trees on those fractal graphs based on Kirchhoff's Matrix-Tree Theorem and the specific knowledge of the spectrum of the probabilistic graph Laplacian on such fractals. The method works due to the so-called spectral decimation property studied in various papers such as [2], [8], [17] among others. Moreover, in [1] the asymptotic complexity constant of such graphs is studied, which is defined as the limit of the logarithm of the number of spanning trees over the number of vertices of the graphs. In [23] the asymptotic complexity constant is studied among others, and precisely evaluated in a closed form for a more general class of self-similar fractal graphs satisfying a symmetry condition called strong symmetry. Here we study it initially without any symmetry conditions and later with a symmetry condition which is called full symmetry. Our approach is based on the spectral decimation property thus viewing the subject from a point of view which is closer to analysis on fractals. It is conjectured in [1] that by adding the assumption of full symmetry we can obtain a proof of the existence of the asymptotic complexity constant using the knowledge of the spectrum of the probabilistic graph Laplacian and based on that, a second conjecture is presented regarding a sharp lower bound of the asymptotic complexity constant. The goal of this paper is to prove these two conjectures (Conjectures 4.3 and 4.4 in [1]) and also obtain an upper bound of the asymptotic complexity constant. Thus we obtain the following result.

Theorem 1.1. *For a given fully symmetric self-similar structure on a finitely ramified fractal K , let G_n denote its sequence of approximating graphs. Then the asymptotic complexity constant exists and if G_0 has more than two vertices we have*

that

$$\frac{\log(3)}{2} \leq c_{asymp} \leq \log \left(\frac{(m-1)|V_0|(|V_0|-1)}{|V_1|-|V_0|} \right)$$

where $|V_0|, |V_1|$ are the number of vertices of the G_0 and G_1 graphs respectively, and m is the number of contractions that create the self similar fractal graph.

Remark. If we consider the m -Tree fractal in [7] we have by Cayley's formula that $\tau(G_0) = m^{m-2}$ and thus we see that $\tau(G_n) = m^{(m-2)m^n}$ and $|V_n| = 1 + (m-1)m^n$ and thus the asymptotic complexity constant is $\frac{(m-2)\log m}{m-1}$. By considering the 3-Tree fractal for $m = 3$ we observe that the asymptotic complexity constant is $\frac{\log 3}{2}$ which means that the lower bound is sharp.

2. BACKGROUND NOTIONS.

First, we present some background material. The discussion here is brief, we refer the reader to [2, 1, 11, 19] for more details. If we have a compact connected metric space (X, d) and $F_i : X \rightarrow X$ are injective contractions for $i = 1, 2, \dots, m$, then there exists a unique non-empty compact subset K of X that satisfies

$$K = F_1(K) \cup \dots \cup F_m(K).$$

and K is called the self-similar set with respect to $\{F_1, F_2, \dots, F_m\}$. If for any two distinct words $w, w' \in W_n = \{1, \dots, m\}^n$ we have that $F_w(K) \cap F_{w'}(K) = F_w(V_0) \cap F_{w'}(V_0)$ then we call K a finitely ramified self similar set. Informally, a finitely ramified self-similar set is such that every cell can be made disconnected with the rest of the set by removing a finite number of points.

For any self-similar finitely ramified set K with respect to $\{F_1, F_2, \dots, F_m\}$, we define the sequence of approximating graphs G_n with vertex set V_n in the following way. For all $n \geq 0$ and for all $\omega \in W_n$ we have that G_0 is defined as the complete graph with vertices V_0 and

$$V_n := \bigcup_{i=1}^m F_i(V_{n-1}) = \bigcup_{w \in W_n} V_w \quad \text{and} \quad G_n := \bigcup_{w \in W_n} G_w,$$

where G_w is the complete graph with vertices V_w and $F_\omega := F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_n}$ for $\omega = w_1 w_2 \dots w_n$. We have $x, y \in V_n$ to be connected with an edge in G_n if $F_i^{-1}(x)$ and $F_i^{-1}(y)$ are connected by an edge in G_{n-1} for some $1 \leq i \leq m$.

A fully symmetric finitely ramified self-similar structure is a self-similar structure K with contractions $\{F_1, F_2, \dots, F_m\}$ such that K is a finitely ramified self-similar set, and for any permutation $\sigma : V_0 \rightarrow V_0$ there is an isometry $g_\sigma : K \rightarrow K$ that maps any $x \in V_0$ into $\sigma(x)$ and preserves the self-similar structure of K . Then we have a map $\tilde{g}_\sigma : W_1 \rightarrow W_1$ such that $F_i \circ g_\sigma = g_\sigma \circ F_{\tilde{g}_\sigma(i)} \forall i \in W_1$.

Then, if G_n for $n \geq 0$ is a sequence of finite graphs, denote $|V_n|$ to be the cardinality of V_n , and $\tau(G_n)$ denote the number of spanning trees of G_n . If the limit

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(G_n))}{|V_n|}$$

exists, it is called the *asymptotic complexity constant* or the *tree entropy* of the sequence G_n . We also have that for any two, finite, connected graphs G_1, G_2 , if

$G_1 \vee_{x_1, x_2} G_2$ denotes the graph formed by identifying the vertex $x_1 \in G_1$ with vertex $x_2 \in G_2$ then $\forall x_1 \in G_1, x_2 \in G_2$, and it is clear that

$$(2.1) \quad \tau(G_1 \vee_{x_1, x_2} G_2) = \tau(G_1) \cdot \tau(G_2).$$

We denote by Δ_n the probabilistic graph Laplacian of G_n which is defined as $\Delta_n = I - T^{-1}A$ where T is the degree matrix and A is the adjacency matrix. The way to study the spectrum of Δ_n is by a process called spectral decimation which recursively computes the spectrum of Δ_{n+1} by using information from the spectrum of Δ_n . It was first studied rigorously on the Sierpinski Gasket by [8] and later it was generalised to other fractals. We refer the reader to [2, 3]. In [2] we have the following propositions which describe the essence of spectral decimation.

We start with V_0 being the complete graph on the boundary set. Write Δ_1 in block form

$$\Delta_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is a square block matrix associated to the boundary points. Since the V_1 network never has an edge joining two boundary points A is the $|V_0| \times |V_0|$ identity matrix. The Schur Complement of Δ_1 is

$$S(z) = (A - zI) - B(D - z)^{-1}C$$

Proposition 2.1. *For a given fully symmetric finitely ramified self-similar structure K there are unique scalar valued rational functions $\phi(z)$ and $R(z)$ such that for $z \notin \sigma(D)$*

$$S(z) = \phi(z)(P_0 - R(z))$$

Now, we let

$$E(\Delta_0, \Delta_1) := \sigma(D) \bigcup \{z : \phi(z) = 0\}$$

and call $E(\Delta_0, \Delta_1)$ the exceptional set.

We denote $\text{mult}_D(z)$ the multiplicity of z as an eigenvalue of D and $\text{mult}_n(z)$ the multiplicity of z as an eigenvalue of Δ_n . In the case that it's not an eigenvalue, we simply say it has multiplicity zero. Then we may inductively find the spectrum of Δ_n with the following proposition.

Proposition 2.2. *If K is a fully symmetric finitely ramified self-similar structure and $R(z)$, $\phi(z)$, $E(\Delta_0, \Delta_1)$ as above, the spectrum of Δ_n may be calculated recursively in the following way:*

- (1) if $z \notin E(\Delta_0, \Delta_1)$, then

$$\text{mult}_n(z) = \text{mult}_{n-1}(R(z))$$

- (2) if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z then,

$$\text{mult}_n(z) = |V_{n-1}|$$

- (3) if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{\partial}{\partial z}R(z) \neq 0$, then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) - |V_{n-1}| + \text{mult}_{n-1}(R(z))$$

- (4) if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) \neq 0$, then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) + \text{mult}_{n-1}(R(z))$$

(5) if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) = 0$, then

$$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) + |V_{n-1}| + \text{mult}_{n-1}(R(z))$$

(6) if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{\partial}{\partial z} R(z) = 0$, then

$$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) - |V_{n-1}| + 2 \text{mult}_{n-1}(R(z))$$

(7) if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole at z , then $\text{mult}_n(z) = 0$.

(8) if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , $\phi(z) = 0$ and $R(z)$ has a pole at z , then

$$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z).$$

As in the notation of [1] it is possible to define sets of eigenvalues A, B and corresponding multiplicities for $\alpha \in A$, $\alpha_n := \text{mult}_n(\alpha)$ and for $\beta \in B$, $\beta_n^k := \text{mult}_n(R_{(-k)}(\beta))$.

Then the spectral decimation algorithm means that the spectrum of the probabilistic graph Laplacian of level n is

$$(2.2) \quad \sigma(\Delta_n) \setminus \{0\} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[\bigcup_{k=0}^n \{R_{-k}(\beta) : \beta_n^k \neq 0\} \right].$$

A famous theorem in graph theory is Kirchhoff's Matrix-Tree Theorem which relates the number of spanning trees of a graph with the product of the eigenvalues of its Laplacian. A version of this theorem for the probabilistic graph Laplacian is the following.

Theorem 2.3 (Kirchhoff's theorem). *For any connected, loopless graph G with n labelled vertices, the number of spanning trees of G is*

$$\tau(G) = \left| \frac{\left(\prod_{j=1}^n d_j \right)}{\left(\sum_{j=1}^n d_j \right)} \prod_{j=1}^{n-1} \lambda_j \right|$$

where λ_j are the non-zero eigenvalues of the probabilistic graph Laplacian.

Kirchhoff's theorem is used in [1] to compute the number of spanning trees of the approximating graphs of a given fully symmetric self-similar structure on a finitely ramified fractal K . Let $R(z) = \frac{P(z)}{Q(z)}$ with $\deg(P(z)) > \deg(Q(z))$, d be the degree of $R(z)$, P_d be the leading coefficient of the numerator of $R(z)$, $|V_n|$ be the number of vertices of G_n and d_j be the degree of vertex j in G_n . Then the number of spanning trees is given by

$$(2.3) \quad \tau(G_n) = \left| \frac{\left(\prod_{j=1}^{|V_n|} d_j \right)}{\left(\sum_{j=1}^{|V_n|} d_j \right)} \left(\prod_{\alpha \in A} \alpha^{\alpha_n} \right) \left[\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \left(\frac{-Q(0)}{P_d} \right)^{\sum_{k=0}^n \beta_n^k \left(\frac{d^k - 1}{d - 1} \right)} \right) \right] \right|.$$

We will also make use of the Stolz-Cesàro lemma which states the following.

Lemma 2.4. *Let $(a_n)_n$ and $(b_n)_n$ be sequences of real numbers such that $(b_n)_n$ is strictly monotone and divergent to $+\infty$ or $-\infty$. If we have that the following limit exists*

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = c,$$

then we have that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$

Now, we are ready to present the proof of the main theorem.

3. PROOF OF THE CONJECTURES.

As before, let G_n be the sequence of approximating graphs, $|V_n|$ the number of its vertices, m the number of 0-cells of the G_1 graph and $\tau(G_n)$ be the number of spanning trees. Due to the self-similarity we have that the G_n graph is m^n copies of the G_0 graph with an appropriate identification of its vertices. In fact, in [23] we have the formula $|V_n| = m|V_{n-1}| - m|V_0| + |V_1|$ from which we can derive that

$$|V_n| = \frac{m^n(|V_1| - |V_0|) + m|V_0| - |V_1|}{m - 1}.$$

Thus we can see that $\lim_{n \rightarrow \infty} \frac{|V_n|}{m^n} = \frac{|V_1| - |V_0|}{m - 1}$. To simplify notation later, we denote this as $|V_n| \sim m^n$.

We give a proof as to why the asymptotic complexity constant is positive and give an upper bound.

Proposition 3.1. *Let K be a self similar, finitely ramified fractal and G_n be its sequence of approximating graphs. If G_1 is not a tree we have that*

$$0 < \liminf_{n \rightarrow \infty} \frac{\log \tau(G_n)}{|V_n|} \leq \limsup_{n \rightarrow \infty} \frac{\log \tau(G_n)}{|V_n|} \leq \log \left(\frac{(m-1)|V_0|(|V_0| - 1)}{|V_1| - |V_0|} \right)$$

Remark. *The first inequality, that it is greater than zero, is proven in [1], we repeat the argument here for the convenience of the reader. We make one modification however. The author asserts that if we assume that G_1 is not a tree then it implies that $|V_0| > 2$. However, this is not the case. There are self similar fractal graphs such that G_0 is the complete graph on two vertices and G_1 is not a tree. Examples of that are the graphs studied in [16] and the so called Austria graphs in [23]. It is interesting then to see what happens in the case that $|V_0| = 2$. Then we may have that G_1 is a tree and thus the number of spanning trees is trivially 1 and therefore the asymptotic complexity constant is 0. However if we assume that G_1 is not a tree, then we will see that a very similar argument to the previous case proves that even if $|V_0| = 2$ we have a positive lower bound.*

Proof. We have that $\tau(G_n) \geq \tau(G_0 \vee_{x,x}^{m^n} G_0)$ where $G_0 \vee_{x,x}^{m^n} G_0$ denotes m^n copies of G_0 each identified to each other at some vertex $x \in V_0$. Assume first that $|V_0| > 2$. Then, since the G_0 graph is the complete graph on $|V_0|$ vertices, by Cayley's formula we have that $\tau(G_0) = |V_0|^{|V_0|-2}$. Thus we get that

$$\tau(G_0 \vee_{x,x}^{m^n} G_0) = |V_0|^{(|V_0|-2) \cdot m^n}$$

and

$$\tau(G_n) \geq |V_0|^{(|V_0|-2) \cdot m^n}.$$

So for $n \geq 0$,

$$(3.1) \quad \log(\tau(G_n)) \geq m^n \cdot (|V_0| - 2) \log(|V_0|)$$

which is equation (6) in [1]. This gives us the result since $|V_n| \sim m^n$. In the case that $|V_0| = 2$ we can say the following. Since G_1 is not a tree, it must be that $m > 2$ and G_1 must contain a cycle and of course $|V_1| > 2$. Then using the same argument as before, we have that G_n is m^{n-1} copies of G_1 with appropriate vertex identification and $\tau(G_n) \geq \tau(G_1 \vee_{x,x}^{m^{n-1}} G_1)$ for some vertex $x \in V_1$. Since G_1 contains a cycle, it must have at minimum 3 spanning trees which gives us that $\tau(G_n) \geq 3^{m^{n-1}}$ and we obtain the lower bound exactly as before.

Now, for the upper bound. First we observe that if we denote EV_n the cardinality of the edge set of G_n then we have that $EV_n = \frac{m^n |V_0| (|V_0| - 1)}{2}$. This can be seen from the self similarity of the graph and the fact that G_0 is the complete graph on V_0 vertices. Also, we have from Kirchhoff's theorem that

$$\log \tau(G_n) = \log \frac{\left(\prod_{j=1}^{|V_n|} d_j \right)}{\left(\sum_{j=1}^{|V_n|} d_j \right)} + \log \prod_{j=1}^{V_n-1} \lambda_j$$

The first summand becomes $\sum_{j=1}^{|V_n|} \log d_j - \log \sum_{j=1}^{|V_n|} d_j$ and by using Jensen's inequality, we obtain that

$$\begin{aligned} \sum_{j=1}^{|V_n|} \log d_j - \log \sum_{j=1}^{|V_n|} d_j &\leq \sum_{j=1}^{|V_n|} \log d_j \leq |V_n| \log \left(\frac{\sum_{j=1}^{|V_n|} d_j}{|V_n|} \right) \\ &= |V_n| \log \frac{2EV_n}{|V_n|} = |V_n| \log \frac{m^n |V_0| (|V_0| - 1)}{|V_n|}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{|V_n|}{m^n} = \frac{|V_1| - |V_0|}{m - 1}$ we get an upper bound for $|V_n|^{-1} \log \frac{\left(\prod_{j=1}^{|V_n|} d_j \right)}{\left(\sum_{j=1}^{|V_n|} d_j \right)}$. Now

for the term $\log \prod_{j=1}^{|V_n|-1} \lambda_j$, we know that the trace of the probabilistic graph Laplacian matrix equals $|V_n|$ and therefore as before

$$\begin{aligned} \sum_{j=1}^{|V_n|-1} \log \lambda_j &\leq (|V_n| - 1) \log \left(\frac{\sum_{j=1}^{|V_n|-1} \lambda_j}{|V_n| - 1} \right) = (|V_n| - 1) \log \frac{|V_n|}{|V_n| - 1} \\ &= \log \left(\frac{|V_n|}{|V_n| - 1} \right)^{|V_n|-1} \rightarrow \log e = 1. \end{aligned}$$

Thus $|V_n|^{-1} \log \prod_{j=1}^{|V_n|-1} \lambda_j \leq 0$ which concludes our proof. \square

We now add the extra assumption of full symmetry and prove the main theorem.

Proof. We want to prove the existence of the limit of the sequence $\frac{\log \tau(G_n)}{|V_n|}$. We already have from the proposition above that the sequence is bounded. Therefore it suffices to check that we do not have any oscillatory behavior. By the full symmetry assumption, we can perform spectral decimation and $\tau(G_n)$ is given by equation (2.3) and thus we obtain that

$$(3.2) \quad \begin{aligned} \log \tau(G_n) = & \log \left| \frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} \right| + \log \left| \prod_{\alpha \in A} \alpha^{\alpha_n} \right| \\ & + \log \left| \prod_{\beta \in B} \beta^{\sum_{k=0}^n \beta_n^k} \right| + \log \left| \prod_{\beta \in B} \left(\frac{-Q(0)}{P_d} \right)^{\sum_{k=0}^n \beta_n^k \frac{d^k-1}{d-1}} \right| \end{aligned}$$

Therefore it suffices to prove that the limit $\lim_{n \rightarrow \infty} \frac{\log \prod_{j=1}^{|V_n|} d_j - \log \sum_{j=1}^{|V_n|} d_j}{|V_n|}$ exists and for each $\alpha \in A$ and $\beta \in B$ the limits

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{|V_n|}, \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \beta_n^k}{|V_n|} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \beta_n^k \frac{d^k-1}{d-1}}{|V_n|}$$

also exist. We know that even though Δ_n is not in general a symmetric operator it has only real eigenvalues. Then, including multiplicities, their total number must be equal to the dimension of the space and thus from equation (2.2) we have the formula

$$\sum_{\alpha \in A} \alpha_n + \sum_{\beta \in B} \sum_{k=0}^n \beta_n^k d^k + 1 = |V_n|.$$

Since α_n, β_n^k are non-negative integers we see that for each $\alpha \in A$ and $\beta \in B$ that $\frac{\alpha_n}{|V_n|}, \frac{\sum_{k=0}^n \beta_n^k d^k}{|V_n|}$ must be bounded and thus the same holds for $\frac{\sum_{k=0}^n \beta_n^k}{|V_n|}$ and $\frac{\sum_{k=0}^n \beta_n^k \frac{d^k-1}{d-1}}{|V_n|}$. Now, for a given $\alpha \in A$, we have that by the definition of the finite set A that the multiplicities $\alpha_n = \text{mult}_n(\alpha)$ which can be found from Proposition 1.3. above depend only on the eigenvalue α and the level n and in each of the cases of the Proposition we have convergence as $|V_n| \sim m^n$. Now for the remaining limits. Take $\beta \in B$ and $\beta_n^k = \text{mult}_n(R_{-k}(\beta))$. By the general algorithm of the spectral decimation methodology, we have that every pre-iterate of the spectral decimation rational function preserves the multiplicity of the eigenvalues. Therefore, we have that $\beta_{n+1}^k = \beta_n^{k-1}$ for $1 \leq k \leq n+1$ and thus the sum of multiplicities at level $n+1$ must be the sum of the multiplicities at level n along with those with generation of birth $n+1$. This is just the following formula

$$\sum_{k=0}^{n+1} \beta_{n+1}^k = \sum_{k=1}^{n+1} \beta_{n+1}^k + \beta_{n+1}^0 = \sum_{k=1}^{n+1} \beta_n^{k-1} + \beta_{n+1}^0 = \sum_{k=0}^n \beta_n^k + \beta_{n+1}^0$$

and

$$\sum_{k=0}^{n+1} \beta_{n+1}^k d^k = \sum_{k=1}^{n+1} \beta_{n+1}^k d^k + \beta_{n+1}^0 = \sum_{k=1}^{n+1} \beta_n^{k-1} d^k + \beta_{n+1}^0 = d \sum_{k=0}^n \beta_n^k d^k + \beta_{n+1}^0$$

By taking into account that $\frac{|V_{n+1}|}{|V_n|} \rightarrow m$ and by looking at the Proposition 1.3. above, we have a list of possible choices for the term β_{n+1}^0 and as similarly to the case of the eigenvalues in the set A before it must be that $\frac{\beta_{n+1}^0}{V_{n+1}}$ converges to a finite positive constant, which we can call c .

For a general first order linear recurrence $S_{n+1} = f_n S_n + g_n$ we know that it has solution

$$S_n = \left(\prod_{k=0}^{n-1} f_k \right) \left(A + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^m f_k} \right)$$

where A is a constant. From the arguments above, we have that $V_{n+1} = y_n V_n$ where y_n is a sequence such that $y_n \rightarrow m$ and $\frac{\beta_{n+1}^0}{V_{n+1}} = c + x_n$ with x_n being a sequence such that $x_n \rightarrow 0$. Then for $S_n = \frac{\sum_{k=0}^n \beta_k^0 d^k}{|V_n|}$ we obtain that $S_{n+1} = \frac{d}{y_n} S_n + c + x_n$. Since we know that S_n is bounded, it must be that $\frac{d}{y_n} \leq 1 - \epsilon$ for some $\epsilon > 0$ and large n . Then

$$S_n = \left(\prod_{k=0}^{n-1} \frac{d}{y_k} \right) \left(A + \sum_{i=0}^{n-1} \frac{c + x_i}{\prod_{k=0}^i \frac{d}{y_k}} \right)$$

We care about the limit of $n \rightarrow \infty$ so the constant part becomes 0 and we are left with

$$cd^n \frac{\sum_{i=0}^{n-1} \prod_{k=0}^i \frac{y_k}{d}}{\prod_{k=0}^{n-1} y_k} + d^n \frac{\sum_{i=0}^{n-1} x_i \prod_{k=0}^i \frac{y_k}{d}}{\prod_{k=0}^{n-1} y_k}$$

The second summand goes to 0 as can be seen by the Stolz-Cesàro lemma in the following way. Due to the fact that $\frac{d}{y_n} \leq 1 - \epsilon$ we have that $\prod_{k=0}^{n-1} \frac{y_k}{d}$ is a strictly increasing sequence diverging to $+\infty$. Then,

$$\frac{\sum_{i=0}^n x_i \prod_{k=0}^i \frac{y_k}{d} - \sum_{i=0}^{n-1} x_i \prod_{k=0}^i \frac{y_k}{d}}{\prod_{k=0}^n \frac{y_k}{d} - \prod_{k=0}^{n-1} \frac{y_k}{d}} = \frac{x_n \prod_{k=0}^n \frac{y_k}{d}}{\prod_{k=0}^{n-1} \frac{y_k}{d} \left(\frac{y_n}{d} - 1 \right)} \rightarrow 0$$

since $y_n \rightarrow m$ and $x_n \rightarrow 0$.

The first summand is just $\sum_{i=0}^{n-1} d^{n-i+1} \prod_{k=i+1}^{n-1} \frac{1}{y_k}$ which is a positive series and since S_n is bounded, it must be that it converges. Thus we get existence of $\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \beta_k^0 d^k}{|V_n|}$. By an exact similar argument, or more easily by the Stolz-Cesàro lemma, we have the existence of the limit $\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \beta_k^0}{|V_n|}$ and thus also we get that $\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \beta_k^0 \frac{d^k - 1}{d - 1}}{|V_n|}$ exists.

We have that $|V_n|^{-1} \log \left| \frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} \right|$ is bounded and that $\lim_{n \rightarrow \infty} \frac{\log \sum_{j=1}^{|V_n|} d_j}{|V_n|} = 0$. More-

over the limit $\lim_{n \rightarrow \infty} \frac{\log \prod_{j=1}^{|V_n|} d_j}{|V_n|}$ cannot oscillate due to the symmetry of the fractal graph and thus obviously exists as it's bounded. Thus all the required limits exist and we obtain our result.

Now, to bound the limit from below. Regarding the number of vertices, we have the following bound

$$|V_n| \leq m^n (|V_0| - 1) + 1.$$

This follows due to the fact that the G_n graph is m^n copies of the G_0 one and therefore we obviously have that $|V_n| \leq m^n |V_0|$. However, due to connectivity, some vertices need to overlap. At minimum, one vertex from each 0-cell will overlap which would mean that

$$|V_n| \leq m^n |V_0| - 1 - 1 - \dots - 1$$

with the number of -1 being as many times as the cells minus one, namely $m^n - 1$ which would give us $|V_n| \leq m^n |V_0| - m^n + 1$.

Then we have the following,

$$\frac{1}{|V_n|} \geq \frac{1}{m^n(|V_0| - 1) + 1} = \frac{1}{m^n(|V_0| - 1 + \frac{1}{m^n})}$$

Then by the above inequality (2.1), we get that

$$\frac{\log(\tau(G_n))}{|V_n|} \geq \frac{m^n(|V_0| - 2) \log(|V_0|)}{m^n(|V_0| - 1 + \frac{1}{m^n})}$$

and thus

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(G_n))}{|V_n|} \geq \frac{(V_0 - 2) \log |V_0|}{|V_0| - 1}.$$

However, since $|V_0|$ is an integer strictly greater than two and we can define the function $f : [3, +\infty] \rightarrow \mathbb{R}$, $f(x) = \frac{(x-2) \log x}{x-1}$ and observe that it has a global minimum at $x = 3$ and therefore

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(G_n))}{|V_n|} \geq \frac{\log 3}{2}.$$

Thus the asymptotic complexity constant must be at least $\frac{\log 3}{2}$ which concludes the proof. \square

Lastly, we illustrate with an example how this methodology can be applied by evaluating the number of spanning trees on the level 3 Sierpinski Gasket. This has been previously evaluated for example in [4], [23] but we reevaluate it here in a different way using the methodology of [1]. For a variety of different examples we refer the reader to [1].

Proposition 3.2. *The number of spanning trees on the level n graph approximation of SG_3 is given for $n \geq 2$ by*

$$\tau(G_n) = 2^{a_n} 3^{b_n} 5^{c_n} 7^{d_n}$$

where $a_n = \frac{2}{5}(6^n - 1)$, $b_n = \frac{1}{25}(13 \cdot 6^n - 15n + 12)$, $c_n = \frac{1}{25}(3 \cdot 6^n - 15n - 3)$, $d_n = \frac{1}{25}(7 \cdot 6^n + 15n - 7)$.

Proof. To calculate the term $\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j}$ it suffices to observe that in SG_3 the boundary points have degree 2, the central ones in each cell have degree 6 and all the rest have degree 4. Moreover, we know that $|V_n| = 3 + \frac{7}{5}(6^n - 1)$. Thus this gives us that

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = \frac{2^3 \cdot 4^{\frac{7}{5}(6^n - 1) - \sum_{k=1}^n 6^{n-k}} \cdot 6^{\sum_{k=1}^n 6^{n-k}}}{4[3 + \frac{7}{5}(6^n - 1)] - 6 + 2 \cdot \sum_{k=1}^n 6^{n-k}} = 2^{-\frac{3}{5} + \frac{13}{5}6^n - n} \cdot 3^{\frac{6^n - 6}{5} - n}$$

The spectrum of Δ_n has been evaluated in [3] using spectral decimation. The rational function of the spectral decimation algorithm is $R(z) = \frac{6z(z-1)(4z-5)(4z-3)}{6z-7}$ and thus $d = 4$, $P_d = 6 \cdot 4^2$ and $Q(0) = -7$. We also have that $A = \{\frac{3}{2}\}$ and $B = \{1, \frac{3}{4}, \frac{5}{4}, \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}\}$. For the corresponding multiplicities we have that for $\alpha = \frac{3}{2}$, $\alpha_n = \frac{2 \cdot 6^n + 8}{5}$. For $\beta_n^k = 1$, $\beta_n^k = 1$. For $\beta = \frac{3}{4}$ or $\beta = \frac{5}{4}$, $\beta_n^k = \frac{3}{5}(6^{n-k-1} - 1)$, $\beta_n^{n-1} = \beta_n^n = 0$. For $\beta = \frac{3+\sqrt{2}}{4}$ or $\beta = \frac{3-\sqrt{2}}{4}$ we have that $\beta_n^k = \frac{2 \cdot 6^{n-k-1} + 8}{5}$, $\beta_n^n = 0$. The eigenvalue 1 has multiplicity one. Then

(3.3)

$$\begin{aligned} \prod_{\alpha \in A} \alpha^{\alpha_n} \prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \cdot \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^n \beta_n^k \frac{4^k - 1}{3}} \right) &= \left(\frac{3}{2} \right)^{\frac{2 \cdot 6^n + 8}{5}} \times \\ &\times \left(\frac{3}{4} \right)^{\sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1)} \times \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1) \frac{4^k - 1}{3}} \\ &\times \left(\frac{5}{4} \right)^{\sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1)} \times \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1) \frac{4^k - 1}{3}} \\ &\times \left(\frac{3 + \sqrt{2}}{4} \right)^{\sum_{k=0}^{n-1} \frac{2 \cdot 6^{n-k-1} + 8}{5}} \times \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^{n-1} \frac{2 \cdot 6^{n-k-1} + 8}{5} \cdot \frac{4^k - 1}{3}} \\ &\times \left(\frac{3 - \sqrt{2}}{4} \right)^{\sum_{k=0}^{n-1} \frac{2 \cdot 6^{n-k-1} + 8}{5}} \times \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^{n-1} \frac{2 \cdot 6^{n-k-1} + 8}{5} \cdot \frac{4^k - 1}{3}} \\ &\times 1 \cdot \left(\frac{7}{4^2 \cdot 6} \right)^{\sum_{k=0}^{n-1} \frac{4^k - 1}{3}} \end{aligned}$$

By calculating the sums above we obtain that

$$\begin{aligned} \sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1) &= \frac{1}{25} (3 \cdot 6^n - 15n - 3) \\ \sum_{k=0}^{n-2} \frac{3}{5} (6^{n-k-1} - 1) \frac{4^k - 1}{3} &= \frac{1}{5} \left(\frac{9}{5} \cdot 6^{n-1} - \frac{10}{3} \cdot 4^{n-1} + n + \frac{8}{15} \right) \\ \sum_{k=0}^{n-1} \frac{2 \cdot 6^{n-k-1} + 8}{5} &= \frac{2}{25} (6^n + 20n - 1) \\ \sum_{k=0}^{n-1} \left(\frac{2 \cdot 6^{n-k-1} + 8}{5} \right) \frac{4^k - 1}{3} &= \frac{6^n - 4^n}{15} + \frac{8(4^n - 1)}{45} - \frac{2}{75} (6^n + 20n - 1) \\ \sum_{k=0}^{n-1} \frac{4^k - 1}{3} &= \frac{4^n - 1}{9} - \frac{n}{3}. \end{aligned}$$

Then by combining those equations above, and doing some elementary calculations we obtain our result. \square

Remark. The asymptotic complexity constant is $C_{\text{asympt}} = \frac{2}{7} \log 2 + \frac{13}{35} \log 3 + \frac{3}{35} \log 5 + \frac{1}{5} \log 7$.

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